Thermal Hall effect of magnons

Hosho Katsura
(Dept. Phys., UTokyo)

Related papers:

- H.K., Nagaosa, Lee, *Phys. Rev. Lett.* **104**, 066403 (2010).
- Onose *et al.*, *Science* **329**, 297 (2010).
- Ideue *et al.*, *Phys. Rev. B* **85**, 134411 (2012).

Outline

1. **Spin Hamiltonian**

- Exchange and DM interactions
- Microscopic origins

2. **Elementary excitations**
3. **Hall effect and thermal Hall effect**
4. **Main results**
5. **Summary**

Coupling between magnetic moments

■ Classical v.s. Quantum

- Dipole-dipole interaction

$$U_{\mathrm{dip}}(\boldsymbol{r}) \propto -\frac{\boldsymbol{\mu}_1 \cdot \boldsymbol{\mu}_2}{r^3} + 3\frac{(\boldsymbol{\mu}_1 \cdot \hat{r})(\boldsymbol{\mu}_2 \cdot \hat{r})}{r^3}$$

Usually, too small (< 1K) to explain transition temperatures…

- Exchange interaction

$$H_{\mathrm{int}} = J\boldsymbol{S}_i \cdot \boldsymbol{S}_j$$

($\boldsymbol{S}_i$: spin at site i)

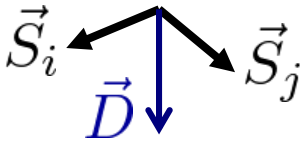
Direct exchange: $J < 0$ → ferromagnetic (FM)
Super-exchange: $J > 0$ → antiferromagnetic (AFM)

■ Anisotropies

Spin-orbit int. breaks SU(2) symmetry.

→ $H_{\mathrm{int}} = J_x S_i^x S_j^x + J_y S_i^y S_j^y + J_z S_i^z S_j^z + \underline{\boldsymbol{D}_{ij} \cdot (\boldsymbol{S}_i \times \boldsymbol{S}_j)} + \cdots$

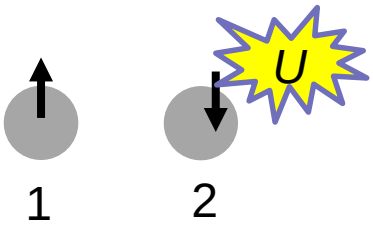
Dzyaloshinskii-Moriya (DM) int.: D
NOTE) Inversion breaking is necessary.



Spin tend to be orthogonal

(Crude) derivation

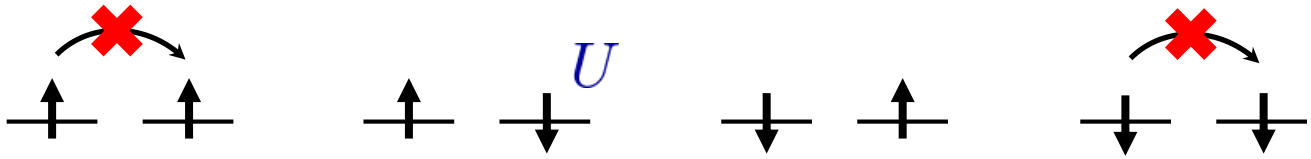
- 2-site Hubbard model
  - Hamiltonian



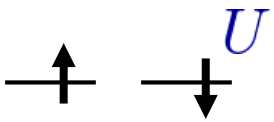
$$H = -t \sum_{\sigma=\uparrow,\downarrow} (c^\dagger_{1,\sigma} c_{2,\sigma} + c^\dagger_{2,\sigma} c_{1,\sigma}) + U \sum_{i=1,2} n_{i,\uparrow} n_{i,\downarrow}$$

  - 2nd order perturbation at half-filling, $U \gg t$

$t = 0$

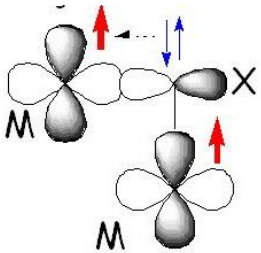


Pauli's exclusion



$$\begin{array}{c} \begin{pmatrix} 0 & & & \\ & -\frac{2t^2}{U} & +\frac{2t^2}{U} & \\ & +\frac{2t^2}{U} & -\frac{2t^2}{U} & \\ & & & 0 \end{pmatrix} \end{array} \begin{array}{c} \uparrow\uparrow \\ \uparrow\downarrow \\ \downarrow\uparrow \\ \downarrow\downarrow \end{array}$$

$$H_{\text{eff}} = \frac{4t^2}{U}\left(\boldsymbol{S}_1 \cdot \boldsymbol{S}_2 - \frac{1}{4}\right) \qquad S_i^\alpha = \frac{1}{2}\sigma_i^\alpha$$



Origin of exchange int. = electron correlation!

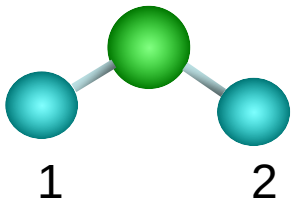
Can explain AFM int. What about FM int.?

(Multi-orbital nature, Kanamori-Goodenough, …)

Origin of DM interaction (1)

■ Spin-dependent hopping

Inversion symmetry broken, but
Time-reversal symmetry exists.



$$H_{\rm hop} = -t \sum_{\sigma=\uparrow,\downarrow} (c^\dagger_{1,\sigma} c_{2,\sigma} + {\rm h.c.})$$

Due to spin-orbit

$$\Rightarrow \quad -\sum_{\sigma,\tau} (c^\dagger_{1,\sigma}\, T_{\sigma,\tau}\, c_{2,\tau} + {\rm h.c.})$$

- Hopping matrix

$$T = t_0 \exp(i\theta\, \boldsymbol{d}\cdot\boldsymbol{\sigma}/2)$$

θ=0 reduces to the spin-independent case

■ Unitary transformation

One can ``absorb" the spin-dependent hopping!

$$c_i = \begin{pmatrix} c_{i,\uparrow} \\ c_{i,\downarrow} \end{pmatrix} \quad (i=1,2) \quad \Rightarrow \quad f_1 = c_1, \quad f_2 = \exp(i\theta\, \boldsymbol{d}\cdot\boldsymbol{\sigma}/2)\, c_2$$

New fermions satisfy the same anti-commutation relations.
Number ops. remain unchanged.

$$n_i = \sum_\sigma c^\dagger_{i,\sigma} c_{i,\sigma} = c^\dagger_i c_i = f^\dagger_i f_i$$

- Hamiltonian in terms of *f*

$$H = H_{\rm hop} + H_U = -t\,(f^\dagger_1 f_2 + {\rm h.c.}) + U \sum_{i=1,2} n_i(n_i - 1)$$

Origin of DM interaction (2)

■ Effective Hamiltonian

$$H_{\rm eff} = \frac{4(t_0)^2}{U}\tilde{\boldsymbol{S}}_1\cdot\tilde{\boldsymbol{S}}_2 \quad \cdots(1) \qquad \tilde{\boldsymbol{S}}_i = f_i^\dagger\frac{\boldsymbol{\sigma}}{2}f_i \quad \left(\boldsymbol{S}_i = c_i^\dagger\frac{\boldsymbol{\sigma}}{2}c_i\right)$$

- How does it look like in original spins?

$$\tilde{\boldsymbol{S}}_1 = \boldsymbol{S}_1, \quad \tilde{\boldsymbol{S}}_2 = c_2^\dagger\, e^{-i\theta\boldsymbol{d}\cdot\boldsymbol{\sigma}/2}\,\frac{\boldsymbol{\sigma}}{2}\,e^{i\theta\boldsymbol{d}\cdot\boldsymbol{\sigma}/2}\,c_2$$
$$= \boldsymbol{S}_2\cos\theta - (\boldsymbol{d}\times\boldsymbol{S}_2)\sin\theta + \boldsymbol{d}\,(\boldsymbol{d}\cdot\boldsymbol{S}_2)(1-\cos\theta)$$

Ex.) Prove the relation.
Hint: express $e^{-i\theta\boldsymbol{d}\cdot\boldsymbol{\sigma}/2}\,\boldsymbol{\sigma}\,e^{i\theta\boldsymbol{d}\cdot\boldsymbol{\sigma}/2}$ in terms of $\boldsymbol{\sigma}$s.

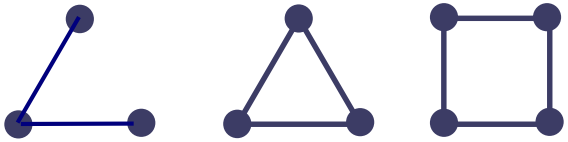
$$(1) = \frac{4t_0^2\cos\theta}{U}\boldsymbol{S}_1\cdot\boldsymbol{S}_2 + \underline{\frac{4t_0^2\sin\theta}{U}\boldsymbol{d}\cdot(\boldsymbol{S}_1\times\boldsymbol{S}_2)} + \frac{4t_0^2(1-\cos\theta)}{U}(\boldsymbol{d}\cdot\boldsymbol{S}_1)(\boldsymbol{d}\cdot\boldsymbol{S}_2)$$

Heisenberg int.

Dzyaloshinskii-Moriya (DM) int.
$\boldsymbol{D}\propto\boldsymbol{d}$

Kaplan-Shekhtman-Aharony -Entin-Wohlman (KSAE) int.

NOTE) One can eliminate the effect of the DM interaction if there is no loop.



Outline

1. Spin Hamiltonian
2. **Elementary excitations**

- What are magnons?
- From spins to bosons
- Diagonalization of BdG Hamiltonian

3. Hall effect and thermal Hall effect
4. Main results
5. Summary

What are magnons?

- FM Heisenberg model in a field

$$H_0 = -J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j - g\mu_{\mathrm{B}} H \sum_i S_i^z \qquad (J > 0)$$

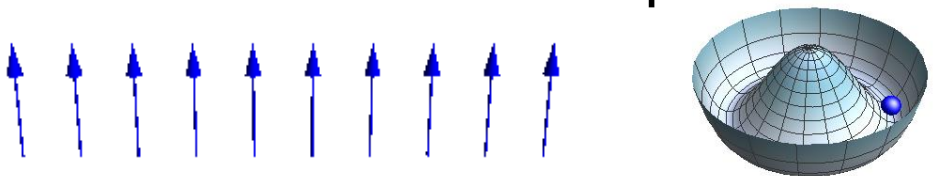
  - Ground state: spins are aligned in the same direction.

$$E_0 = -N(zJS^2 + g\mu_B HS)$$

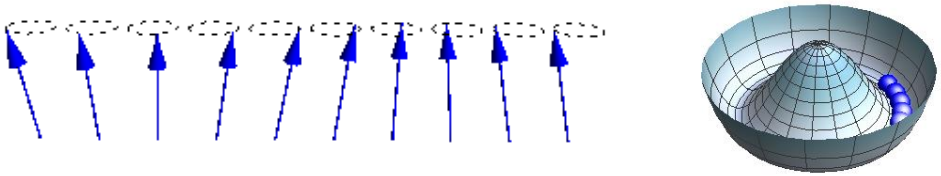
z: coordination number

- Elementary excitations -- Intuitive picture --

Ground state



Excitation =NG mode



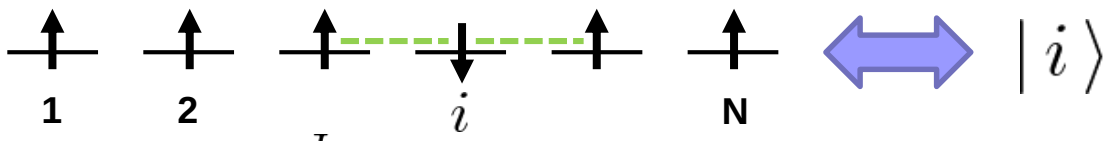
$$E(k) \propto k^2$$

Cf.) non-relativistic Nambu-Goldstone bosons
Watanabe-Murayama, *PRL* **108** (2012); Hidaka, *PRL* **110** (2013).

The picture is classical. But in ferromagnets, ground state and 1-magnon states are *exact* eigenstates of the Hamiltonian.

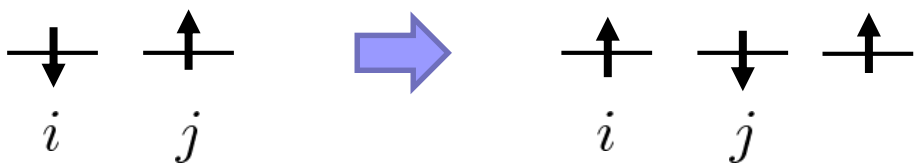
1-Magnon eigenstates

- ``Motion" of flipped spin



|*i*> is *not* an eigenstate!

$$-J\vec{S}_i\cdot\vec{S}_j = -\frac{J}{2}(\underline{S_i^+S_j^-} + S_i^-S_j^+) + JS_i^zS_j^z$$



Flipped spin hops to the neighboring sites.

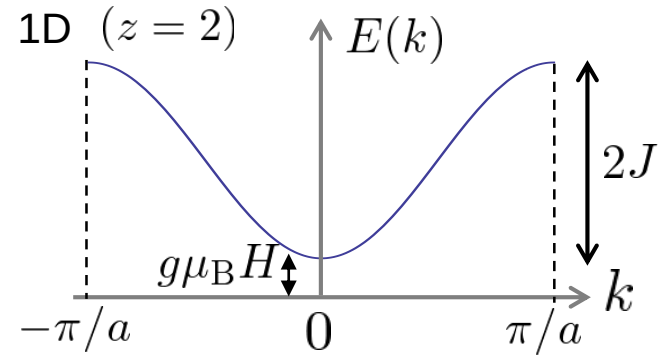
$$-J(\vec{S}_i\cdot\vec{S}_j)|\,i\,\rangle = -\frac{J}{2}|\,j\,\rangle + \frac{J}{4}|i\rangle$$

- Bloch state

$$|\,\psi(\vec{k})\,\rangle = \frac{1}{\sqrt{N}}\sum_i e^{i\vec{k}\cdot\vec{R}_i}|\,i\,\rangle$$

is an exact eigenstate with energy *E*(*k*)

Ex.) 1D $(z=2)$



- What about DM int.?

D vector // *z*-axis

$$-J\vec{S}_i\cdot\vec{S}_j + D[\vec{S}_i\times\vec{S}_j]_z \qquad (\tan\phi = D/J)$$

$$= -\frac{\tilde{J}}{2}(e^{-\mathrm{i}\phi}S_i^+S_j^- + e^{\mathrm{i}\phi}S_i^-S_j^+) - JS_i^zS_j^z$$

Magnon picks up a phase factor!

$$|\,i\,\rangle \to -\frac{\tilde{J}}{2}e^{-\mathrm{i}\phi}|\,j\,\rangle$$

From spins to bosons

■ Holstein-Primakoff transformation

- Bose operators

$$[b_i, b_j^\dagger] = \delta_{ij}, \quad [b_i, b_j] = [b_i^\dagger, b_j^\dagger] = 0$$

Number op.: $n_i = b_i^\dagger b_i$

- Spins in terms of *b*

$$S_j^+ = \sqrt{2S - n_j}\, b_j, \quad S_j^- = b_j^\dagger \sqrt{2S - n_j}, \quad S_j^z = S - n_j$$

Obey the commutation relations of spins

Often neglect nonlinear terms. $S_j^+ \sim \sqrt{2S}\, b_j, \quad S_j^- \sim \sqrt{2S}\, b_j^\dagger$

(Good at low temperatures.)

b raises S^z

- Magnetic ground state = vacuum of bosons $b_j|\mathrm{vac}\rangle = 0$

■ Sublattice structure

AFM int. → Approximate 1-magnon state

- Spins on the other sublattice:

$$S_j^+ = a_j^\dagger \sqrt{2S - n_j}, \quad S_j^- = (S_j^+)^\dagger, \quad S_j^z = -S + n_j$$

a lowers S^z

One needs to introduce more species for a more complex order.

Diagonalization of Hamiltonian

■ Quadratic form of bosons

$$H_{\rm mag} = \frac{1}{2}(\boldsymbol{b}^\dagger, \boldsymbol{b}) \underbrace{\begin{pmatrix} h & \Delta \\ \Delta^* & h^{\rm T} \end{pmatrix}}_{\mathcal{H}} \begin{pmatrix} \boldsymbol{b} \\ \boldsymbol{b}^\dagger \end{pmatrix}$$

$\boldsymbol{b} = (b_1, b_2, ..., b_N)^{\rm T}$

h, Δ : $N \times N$ matrices

$h^\dagger = h,\ \Delta^{\rm T} = \Delta$

- Ferromagnetic case

$$\Delta = \Delta^* = 0 \quad \Rightarrow \quad H_{\rm mag} = \boldsymbol{b}^\dagger h\, \boldsymbol{b} + {\rm const.}$$

Problem reduces to the diagonalization of *h*.
Most easily done in *k*-space (Fourier tr.).

- AFM (or more general) case

$$\begin{pmatrix} \boldsymbol{b} \\ \boldsymbol{b}^\dagger \end{pmatrix} = \mathcal{T} \begin{pmatrix} \boldsymbol{\beta} \\ \boldsymbol{\beta}^\dagger \end{pmatrix}$$

Para-unitary

$$\mathcal{T}\sigma_3\mathcal{T}^\dagger = \mathcal{T}^\dagger\sigma_3\mathcal{T} = \sigma_3 \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Transformation leaves the boson commutations unchanged.

$$\mathcal{T}^\dagger\mathcal{H}\mathcal{T} = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} \qquad E = {\rm diag}\,(\epsilon_1, \epsilon_2, ..., \epsilon_n)$$

$\pm\epsilon_k$ are e.v. of $\sigma_3\mathcal{H}$

Involved procedure. See, e.g., Colpa, *Physica* **93A**, 327 (1978).

Outline

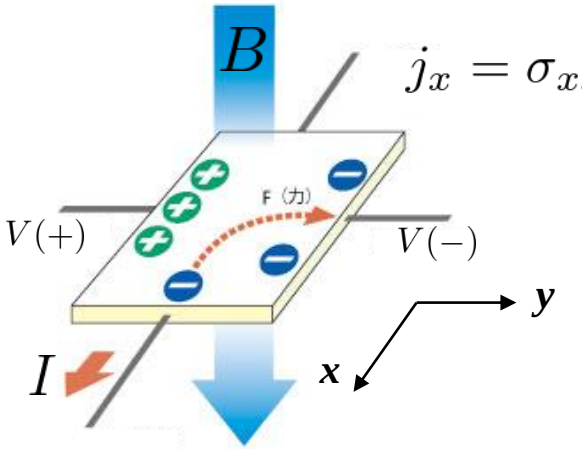
1. Spin Hamiltonian
2. Elementary excitations
3. **Hall effect and thermal Hall effect**

- Hall effect and Berry curvature
- Anomalous and thermal Hall effects
- General formulation

4. Main results
5. Summary

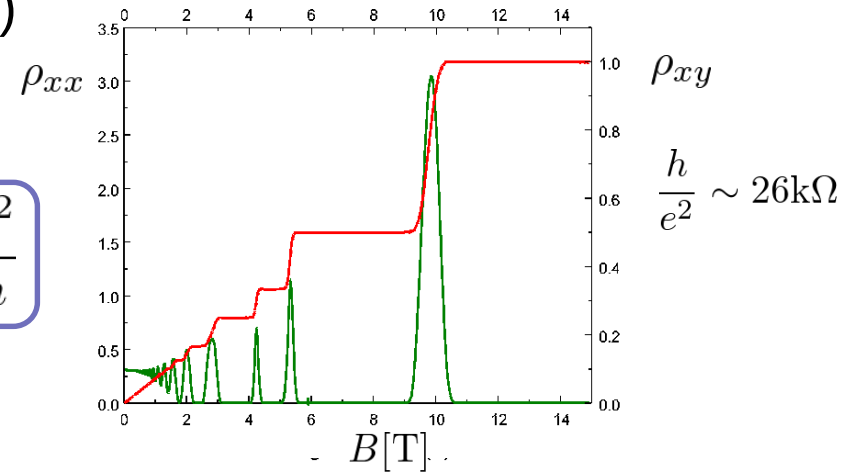
Hall effect and Berry curvature

■ Quantum Hall effect (2D el. Gas)



$$j_x = \sigma_{xx} E_x + \sigma_{xy} E_y$$

$$\sigma_{xy} = N \frac{e^2}{h}$$



■ TKNN formula *PRL*, **49** (1982)

Integer *n* is a topological number!

- Bloch wave function $\psi_n(\boldsymbol{k})$
- Berry connection $a_\alpha^{(n)}(\boldsymbol{k}) = i \langle \psi_n(\boldsymbol{k}), \frac{\partial}{\partial k_\alpha} \psi_n(\boldsymbol{k}) \rangle \qquad (\alpha = x, y, z)$
- Berry curvature $F_{\alpha,\beta}^{(n)}(\boldsymbol{k}) = \partial_\alpha a_\beta^{(n)}(\boldsymbol{k}) - \partial_\beta a_\alpha^{(n)}(\boldsymbol{k})$

Chern number

$$N = -\sum_{n \in \mathrm{occ.}} \int_{\mathrm{BZ}} \frac{d^2\boldsymbol{k}}{(2\pi)^2} F_{xy}(\boldsymbol{k})$$

Kubo formula relates Chern # and σ_{xy}

Anomalous Hall effect

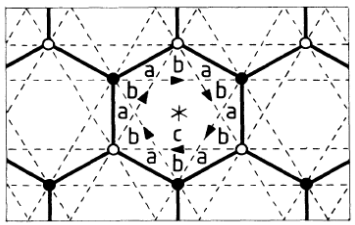
■ QHE without net magnetic field

- Onsager's reciprocal relation

$$\sigma_{xy}(H) = -\sigma_{xy}(-H)$$

Time-reversal symmetry (TRS) must be broken for nonzero σ_{xy}

- Haldane's model (*PRL* **61**, 2015 (1988), Nobel prize 2016)



Local magnetic field can break TRS!

n.n. real and n.n.n. complex hopping
→ Integer QHE without Landau levels

■ Spontaneous symmetry breaking

TRS can be broken by magnetic ordering.

- Anomalous Hall effect Review: Nagaosa *et al*., *RMP* **82**, 1539 (2010).

$$\rho_{xy} = R_0 H_z + R_s M_z$$

M_z : magnetization

Itinerant electrons in ferromagnets.
(i) Intrinsic and (ii) extrinsic origins.
Anomalous velocity by Berry curvature in (i). $\vec{v}_{\rm an} \propto \vec{E} \times \vec{b}_n(\vec{k})$

Thermal Hall effect

■ Thermal current

$$\boldsymbol{J} = C_{11}\boldsymbol{E} + C_{12}(-\nabla T) \qquad (C_{11} = \sigma)$$
$$\boldsymbol{J}_E = C_{21}\boldsymbol{E} + C_{22}(-\nabla T)$$

Cs are matrix, in general.

Onsager relation: $C_{21} = TC_{12}$

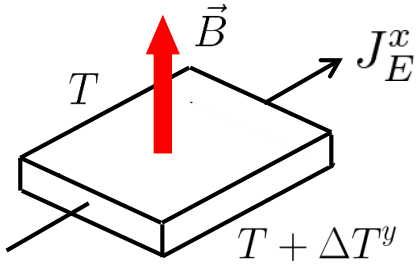
Absence of $\boldsymbol{J}$: $\boldsymbol{E} = (C_{12}/C_{22})\nabla T$ ➡ $\kappa = C_{22} - \dfrac{C_{12}C_{21}}{C_{11}}$

- Wiedemann-Franz law

$$\frac{\kappa}{\sigma T} = L := \frac{\pi^2}{3}\frac{k_{\mathrm{B}}^2}{e^2} = 2.45 \times 10^{-8}[\mathrm{W\Omega/K^2}]$$

Universal for weakly interacting electrons

■ Righi-Leduc effect



Transverse temperature gradient is produced in response to heat current

$$\kappa_{xy} = LT\sigma_{xy}$$

In itinerant electron systems from Wiedemann-Franz

What about Mott insulators? Hall effect without Lorentz force?

← Berry curvature plays the role of magnetic field!

General formulation

■ TKNN-like formula for bosons

- Bloch w.f. $\psi_n(\boldsymbol{k})$, Berry curvature $F^{(n)}_{\alpha\beta}(\boldsymbol{k})$

Still well defined for 1-magnon Hamiltonian without paring term

- Earlier work
  - Fujimoto, *PRL* **103**, 047203 (2009)
  - H.K., Nagaosa & Lee, *PRL* **104**, 066403 (2010)
  - Onose *et al*., *Science* **329**, 297 (2010)

$$\kappa_{xy} \sim \frac{\Delta^2}{4T} \int_{\mathrm{BZ}} \frac{d^3\boldsymbol{k}}{(2\pi)^3}\, \rho_1(\boldsymbol{k})\, F^{(1)}_{xy}(\boldsymbol{k})$$

Δ: energy separation

$$\rho_n(\boldsymbol{k}) = \frac{1}{e^{\beta\omega_n(\boldsymbol{k})} - 1}$$

Bose distribution

Terms due to the orbital motion of magnon are missing…

- Modified linear-response theory
  - Matsumoto & Murakami, *PRL* **106**, 197202; *PRB* **84**, 184406 (2011)

$$\kappa_{xy} = -\frac{k_B^2 T}{\hbar} \sum_n \int_{\mathrm{BZ}} \frac{d^2\boldsymbol{k}}{(2\pi)^3}\, c_2(\rho_n(\boldsymbol{k}))\, F^{(n)}_{xy}(\boldsymbol{k})$$

$$c_2(\rho) = (1+\rho)\left(\ln\frac{1+\rho}{\rho}\right) - (\ln\rho)^2 - 2\mathrm{Li}_2(-\rho)$$

Universally applicable to (free) bosonic systems!

Magnons, phonons, triplons, photons (?) … NOTE) No quantization.

Outline

1. Spin Hamiltonian
2. Elementary excitations
3. Hall effect and thermal Hall effect
4. **Main results**

- Kagome-lattice FM
- Pyrochlore FM
- Comparison of theory and experimement

5. Summary

Magnon Hall effect

■ Theory

Magnons do not have charge. They do not feel Lorentz force.
Nevertheless, they exhibit thermal Hall effect (THE)!

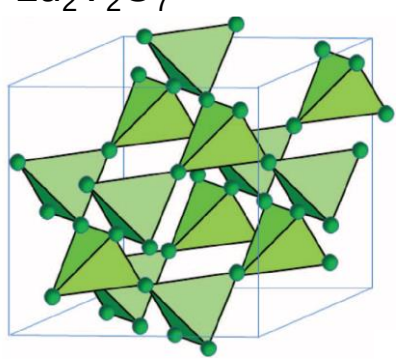
Keys:
1. TRS is broken spontaneously in FM
2. DM interaction leads to Berry curvature ≠ 0

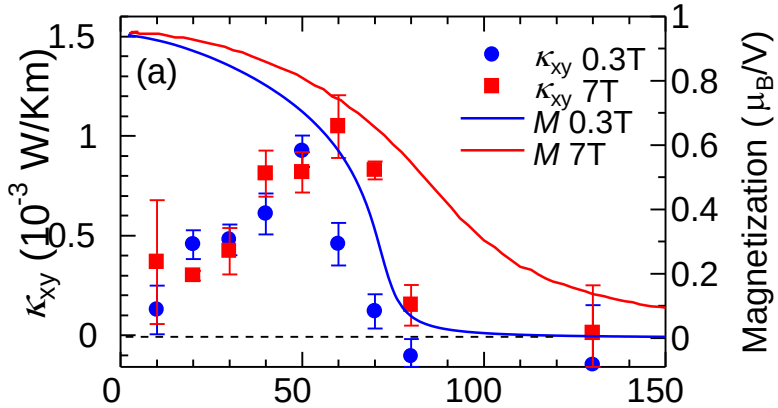
NOTE) Original theory concerned the effect of scalar chirality.

■ Experiment

Magnon THE was indeed observed in FM insulators!
Onose *et al*., *Science* **329**, 297 (2010).

$Lu_2V_2O_7$





Role of DM interaction

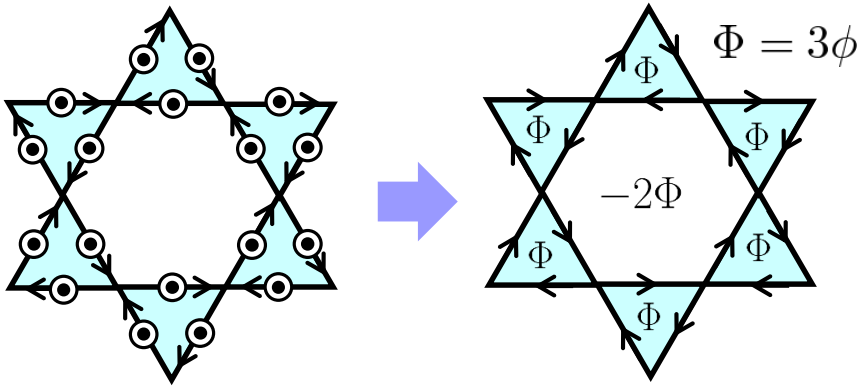
■ Kagome model

$$H = \sum_{\langle i,j \rangle} [\underline{-J\boldsymbol{S}_i \cdot \boldsymbol{S}_j + \boldsymbol{D}_{ij} \cdot (\boldsymbol{S}_i \times \boldsymbol{S}_j)}]$$

$$-\tilde{J}S(e^{i\phi} b_i^\dagger b_j + e^{-i\phi} b_j^\dagger b_i) + \cdots$$

$$\tilde{J} = \sqrt{J^2 + D^2}, \quad \tan\phi = D/J$$

DM vectors

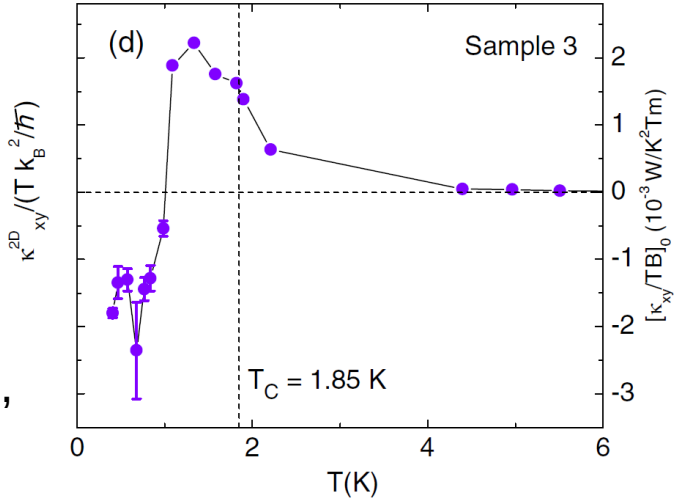


Bosonic ver. of Ohgushi-Murakami-Nagaosa (*PRB* **62** (2000))
Scalar chirality order there ($\boldsymbol{S}_i \cdot (\boldsymbol{S}_j \times \boldsymbol{S}_k)$) ←→ DM.
Nonzero Berry curvature! κ_{xy} is expected to be nonzero.

■ MOF material Cu(1-3, bdc)

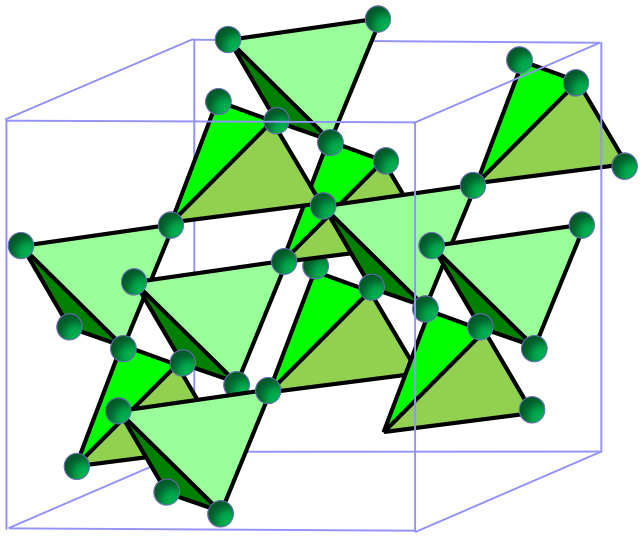
FM exchange int. b/w Cu^{2+} moments
- Hirschberger *et al*., *PRL* **115**, 106603 (2015)
- Chisnell et al., *PRL* **115**, 147201 (2015)

Nonzero THE response.
Sign change consistent with theories:
Mook, Heng & Mertig *PRB* **89**, 134409 (2014),
Lee, Han & Lee, *PRB* **91**, 125413 (2015).



Pyrochlore ferromagnet $Lu_2V_2O_7$

Y. Onose *et al.*, *Science* **329**, 297 ('10).



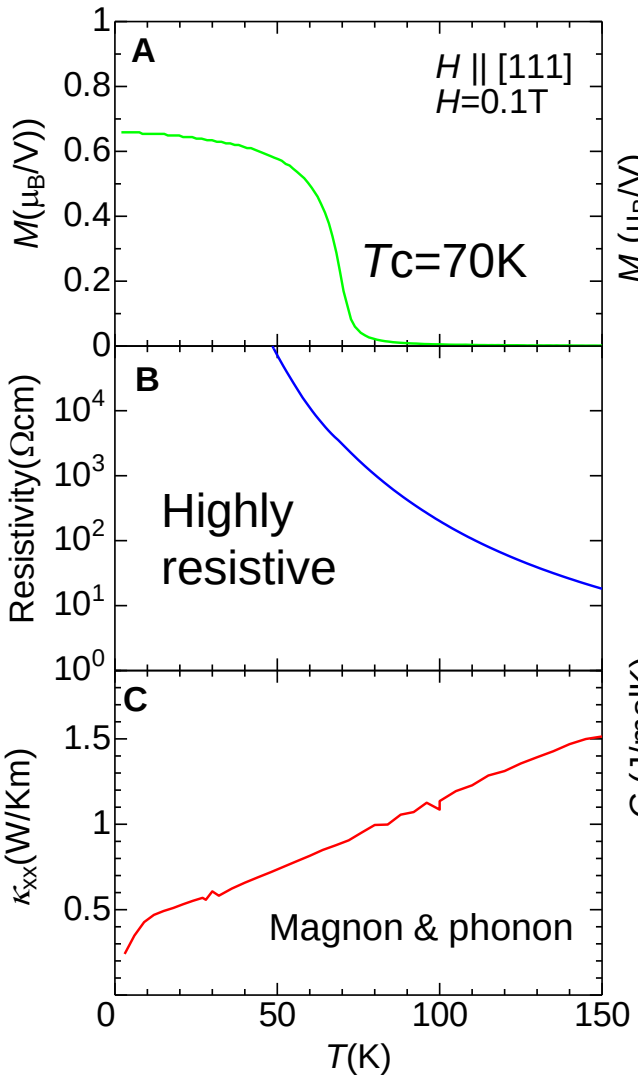
- V^{4+}: $(t_{2g})^1$, S=1/2

- Trigonal crystal field

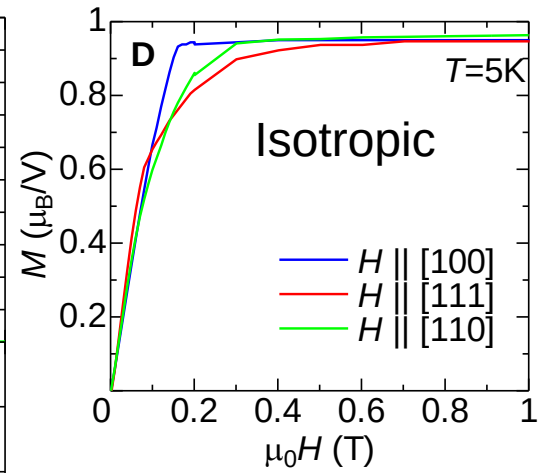
$$|a_{1g}\rangle = \frac{1}{\sqrt{3}}(|xy\rangle + |yz\rangle + |zx\rangle)$$

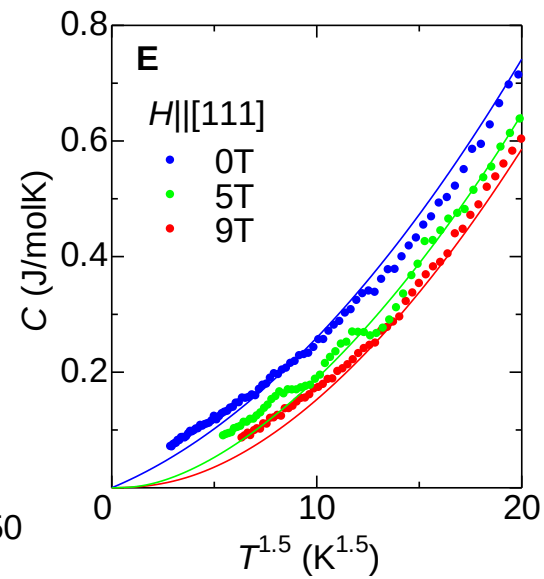
- Origin of FM: orbital pattern

Polarized neutron diffraction (Ichikawa et al., *JPSJ* **74** ('03))



$$\kappa_{xx}^{\mathrm{el}} \sim 10^{-5}\mathrm{W/Km}$$

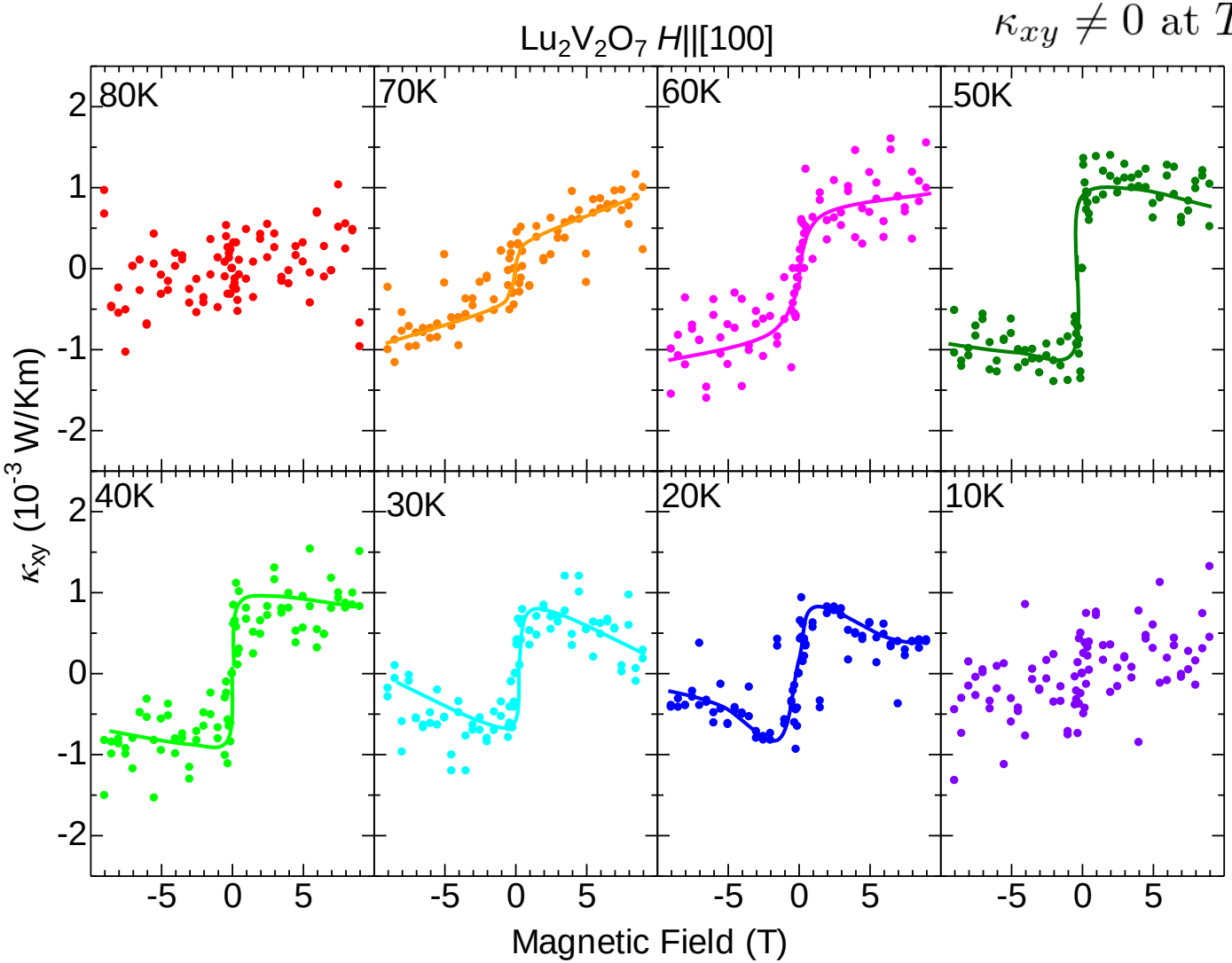




$$C_{\mathrm{ph}} + C_{\mathrm{mag}}$$

Observed thermal Hall conductivity

$\kappa_{xy} \neq 0$ at $T < T_C$

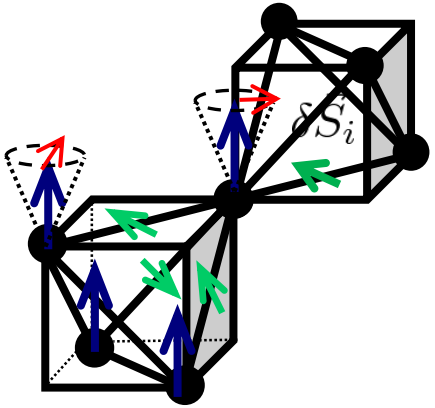


$\kappa_{xy} \neq 0$ at $H \to 0^+$ Anomalous? Related to TRS breaking?

Model Hamiltonian

■ FM Heisenberg + DM

$$H = \sum_{\langle i,j \rangle} [-J\boldsymbol{S}_i \cdot \boldsymbol{S}_j + \boldsymbol{D}_{ij} \cdot (\boldsymbol{S}_i \times \boldsymbol{S}_j)] - \boldsymbol{H} \cdot \sum_i \boldsymbol{S}_i$$



- Allowed DM vectors

  Elhajal *et al.*, *PRB* **71**; Kotov *et al.*, *PRB* **72** (2005).

- Stability of FM g.s. against DM

■ Spin-wave Hamiltonian

Only $\boldsymbol{D} \parallel \boldsymbol{H}$ is important. $\because \delta\boldsymbol{S}_i \times \delta\boldsymbol{S}_j \parallel \boldsymbol{H}$

$$H_{\text{eff}} = \sum_{\langle i,j \rangle} [\underline{-J\boldsymbol{S}_i \cdot \boldsymbol{S}_j + (\boldsymbol{D}_{ij} \cdot \hat{\zeta})(\boldsymbol{S}_i \times \boldsymbol{S}_j)^{\zeta}}] - H \sum_i S_i^{\zeta}$$

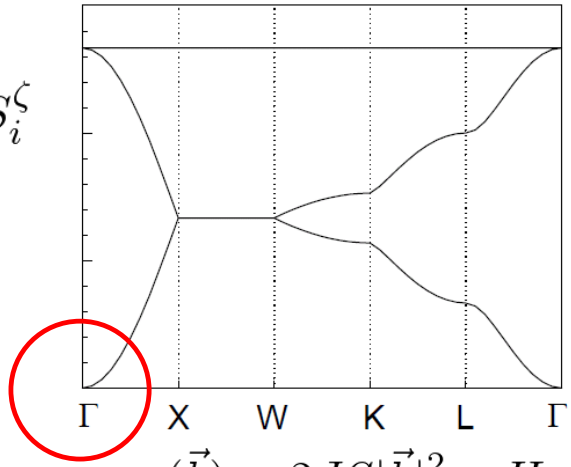
$$-J_{ij} S (e^{-i\phi_{ij}} b_i^\dagger b_j + e^{i\phi_{ij}} b_i^\dagger b_j) + \ldots$$

- Hamiltonian in k-space

$$H_{\text{mag}}(\boldsymbol{k}) = -2JS\,\Lambda(\boldsymbol{k}, \{\phi_{ij}\}) + \text{const.}$$

Λ: 4x4 matrix. → 4 bands

Band structure ($\phi = 0$)



$$\omega_1(\vec{k}) \sim 2JS|\vec{k}|^2 + H$$

Comparison of theory and experiment

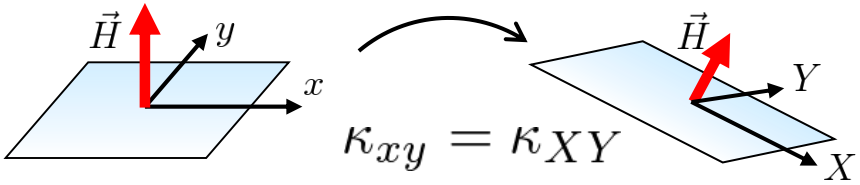
■ Formula (at H=0$^+$)

$$\kappa_{\alpha\beta} = \Phi_{\alpha\beta}\,\frac{4k_B^2 T}{3\pi^2\hbar a}\left(\frac{k_B T}{2JS}\right)^{5/2}\int_0^\infty c_2[(e^t-1)^{-1}]\,t^{3/2}\,dt$$

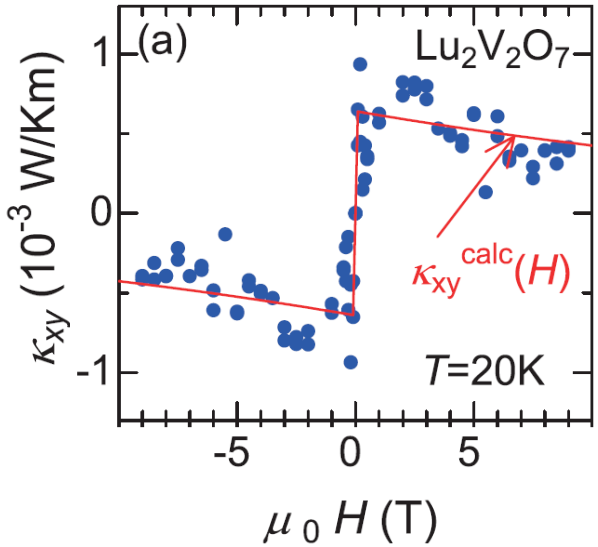
$$\Phi_{\alpha\beta} = \frac{D}{8\sqrt{2}\,J}\,\epsilon_{\alpha\beta\gamma}\,n_\gamma \quad (\alpha,\beta,\gamma = x,y,z)$$

Berry curvature around k=0 can be obtained analytically

- Explains the observed isotropy
- D/J is the only fitting parameter



■ Fitting



The fit yields $|D/J|$ ~ 0.38

Observed in other pyrochlore FM insulators
$Ho_2V_2O_7$: D/J ~ 0.07, $In_2Mn_2O_7$: D/J ~ -0.02

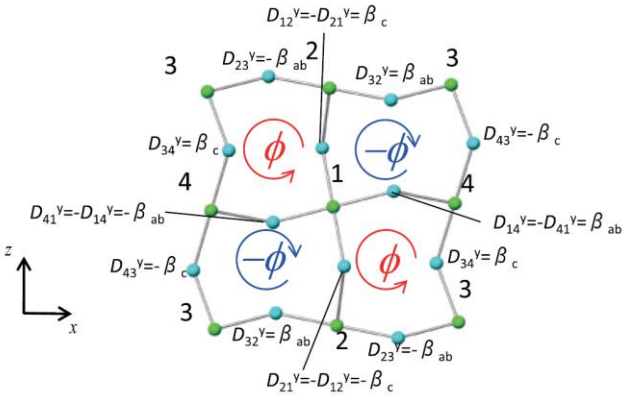
Reasonable!

Cf.) D/J ~ 0.19 in pyrochlore AFM $CdCr_2O_4$

Chern, Fennie & Tchernyshov, *PRB* **74** (2006).

What about other lattices

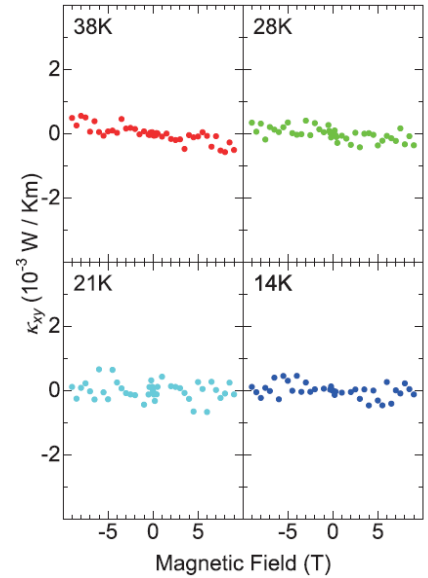
- Provskite-like lattices
  - Absence of THE in La_2NiMnO_6 and $YTiO_3$
    - Ideal cubic perovskite → No DM
    - In reality, it's distorted → nonzero DM



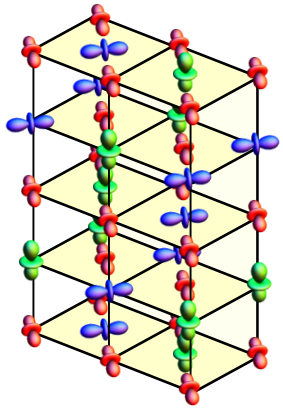
What's the reason?

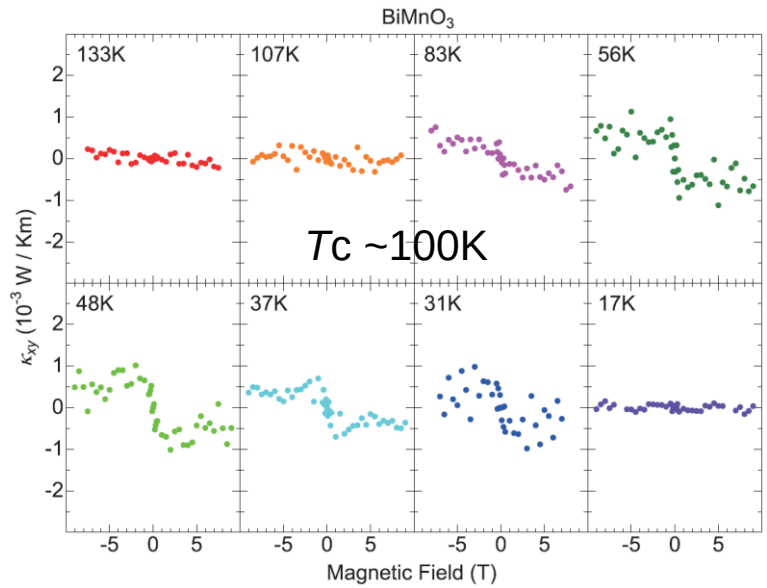
Flux pattern → staggered Berry curvature is zero because of pseudo TRS in

$YTiO_3$, S=1/2, Tc=30K



- Presence of THE in $BiMnO_3$
  - The origin is unclear…
    May be due to complex orbital order

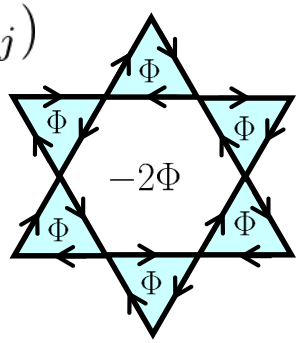




Summary

■ Thermal Hall effect in FM insulators

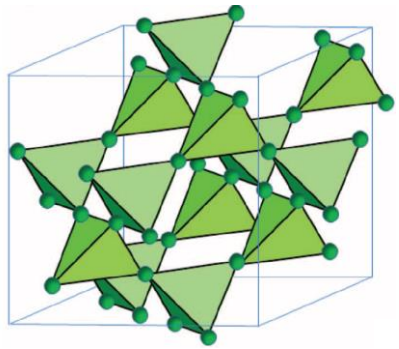
- Mechanism
  Heat current is carried by magnons.
  Driven by Berry curvature due to DM int. $\boldsymbol{D}\cdot(\boldsymbol{S}_i\times\boldsymbol{S}_j)$

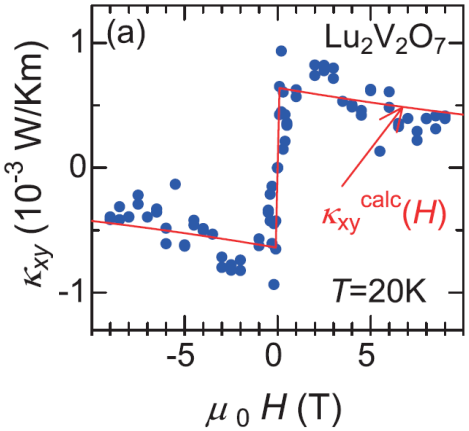


- TKNN-like formula

$$\kappa_{xy} = -\frac{k_B^2 T}{\hbar}\sum_n \int_{\mathrm{BZ}} \frac{d^2\boldsymbol{k}}{(2\pi)^3}\, c_2(\rho_n(\boldsymbol{k}))\, F_{xy}^{(n)}(\boldsymbol{k})$$

- Observation in pyrochlore FMs
  $Lu_2V_2O_7$, $Ho_2V_2O_7$, $In_2Mn_2O_7$





Consistency
- Below FM transition
- Isotropy of κ_{xy}
- Reasonable *D*/*J*

Agreement is excellent!

Mysteries
- Nonzero κ_{xy} in $BiMnO_3$
- Effect of int. between magnons

Other directions

■ Thermal Hall effects of bosonic particles

- Phonon:
  (Exp.) Strohm, Rikken, Wyder, *PRL* **95**, 155901 (2005)
  (Theory) Sheng, Sheng, Ting, *PRL* **96**, 155901 (2006)
  Kagan, Maksimov, *PRL* **100**, 145902 (2008)
- Triplon: (Theory) Romhányi, Penc, Ganesh, *Nat. Comm*. **6**, 6805 (2015)
- Photon: (Theory) Ben-Abdallah, *PRL* **116**, 084301 (2016)

■ Topological magnon physics

- Dirac magnon
  Honeycomb: Fransson, Black-Schaffer, Balatsky, *PRB* **94**, 075401 (2016)
- Weyl magnon
  Pyrochlore AFM: F-Y. Li *et al*., *Nat. Comm.* **7**, 12691 (2016)
  Pyrochlore FM: Mook, Henk, Mertig, *PRL* **117**, 157204 (2016)
- Topological magnon insulators
  Nakata, Kim, Klinovaja, Loss, *PRB* **96**, 224414 (2017)